COMPLEX MARTINGALE CONVERGENCE*

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ABSTRACT

We investigate martingales appropriate for use in complex Banach spaces in connection with the complex uniform convexity popularized by Davis, Garling and Tomczak. This brings us into contact with diverse concepts, such as: pseudo-convex sets, plurisubharmonic functions, conformal martingales, the Radon-Nikodym property, and the analytic Radon-Nikodym property.

0. Introduction

Martingales with values in a Banach space have been used to study the structure of Banach spaces. One example is the equivalence of convergence of L^1-bounded martingales and the Radon-Nikodym property [7]. There are many other examples but convergence is the only one explicitly considered here.

When a Banach space has complex scalars (as opposed to real scalars), the techniques used for its study may have to take that into account. I will be concerned here with martingales that are useful in spaces over the complex numbers.

Here is a simple example of what is involved, taken from a paper by Davis, Garling and Tomczak [6]. Let $\Omega = [0,1[^I$, with product measure. If $\omega \in \Omega$, write $\omega_1, \omega_2, \ldots$ for its components. Define the sequence (X_n) of random variables by:

$$(*) \qquad X_n(\omega) = \sum_{k=1}^{n} f_k(\omega_1, \ldots, \omega_{k-1}) e^{2\pi i \omega_k}$$

(where the f_n's are measurable and bounded). Then (X_n) is a martingale of a very special form. If the f_n's have values in a Banach space E, then (X_n) is a martingale in E. For example, take $E = L^1([0,1])$. Then E fails Radon-Nikodym property. So there are L^1-bounded martingales in E that diverge. But in fact all L^1-bounded martingales in E

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of the form (*) converge a.s. (see [2], Corollary 4.3). This paper arose as an attempt to understand this fact.

The martingales (*) form an interesting class, but they are too limited for many purposes. For example, this class is not closed under optional sampling: if $\tau_1 \le \tau_2 \le \tau_3 \le \dots$ is a sequence of (bounded) stopping times, $Y_n = X_{\tau_n}$ is a new martingale. But if (X_n) is of the form (*), it does not follow that (Y_n) is of the form (*). In this paper I am concerned with how the definition should be extended to have more useful permanence properties, but still to retain the convergence properties exhibited by (*).

The concepts that are used include some from the field of several complex variables (as might be expected, especially when the Banach spaces are finite-dimensional). But care must be taken in how the concepts are formulated. Generally, derivatives must be avoided; many of the Banach spaces of interest do not admit equivalent differentiable norms. So, for example, there is no way to define "the unit ball is strictly pseudo-convex" using derivatives.

1. Preliminaries

We begin with the finite-dimensional definitions. Let U be an open set in $\mathbb{C}$. A function $\psi : U \to [-\infty, \infty[$ is called <u>subharmonic</u> on U iff ψ is upper semicontinuous and, for all $x \in U$ and all $y \in \mathbb{C}$, if $\{x + \lambda y : |\lambda| \le 1\} \subseteq U$, then

$$\int_0^{2\pi} \psi(x + e^{i\theta}y) \frac{d\theta}{2\pi} \ge \psi(x).$$

Note that we have allowed $\psi(x) \equiv -\infty$.

Let U be an open set in $\mathbb{C}^n$, where n is a positive integer. A function $\psi : U \to [-\infty, \infty[$ is called <u>plurisubharmonic</u> iff its restriction to each complex line in $\mathbb{C}^n$ is subharmonic. That is, if $x, y \in \mathbb{C}^n$, and $\theta : \mathbb{C} \to \mathbb{C}^n$ is defined by $\theta(\lambda) = x + \lambda y$, then $\psi \circ \theta$ is subharmonic on $\theta^{-1}[U]$.

Let U be an open set in $\mathbb{C}^n$. Then U is called <u>pseudoconvex</u> iff the function $\psi : U \to [-\infty, \infty[$ defined by

$$\psi(x) = -\log \operatorname{dist}(x, \mathbb{C}^n \setminus U)$$

is plurisubharmonic on U. We intend the distance in the Euclidean norm on $\mathbb{C}^n$, but an equivalent definition is obtained if any other norm on $\mathbb{C}^n$ is substituted.

Discussion of plurisubharmonic functions and pseudoconvex regions in $\mathbb{C}^n$ can be found in many text on several complex variables. One discussion, which emphasizes the similarity to convex functions and convex regions in $\mathbb{R}^n$, has been given by Bremermann [3].

It is natural to extend the definitions to infinite-dimensional spaces. (See, for example, [13]).

Let E be a complex topological vector space. Let U be an open set in E. Then U is said to be <u>pseudoconvex</u> iff $U \cap F$ is pseudoconvex for every finite-dimensional subspace F of E. A function $\psi : U \to [-\infty, \infty[$ is called <u>plurisubharmonic</u> iff ψ is upper semicontinuous and its restriction to each complex line in E is subharmonic. The last part can be rephrased as follows: If $x, y \in E$ and $\{x + \lambda y : |\lambda| \leq 1\} \subseteq U$, then

$$\int_0^{2\pi} \psi\,(x + e^{i\theta} y)\, \frac{d\theta}{2\pi} \geq \psi\,(x).$$

A topological vector space is called <u>locally pseudoconvex</u> (or locally holomorphic [1]) iff there is a base of balanced pseudoconvex neighborhoods of the origin. (See also [1], [14].)

If U is an open balanced neighborhood of 0 and ϕ is its Minkowski gauge, then the following are equivalent (see [1, p.40]):

(a) U is pseudoconvex;

(b) ϕ is plurisubharmonic;

(c) $\log \phi$ is plurisubharmonic.

A <u>quasi-norm</u> on E is a function $\phi : E \to [0, \infty[$ satisfying: $\phi(\lambda x) = |\lambda| \phi(x)$ for $x \in E$, $\lambda \in \mathbb{C}$; $\phi(x + y) \leq K(\phi(x) + \phi(y))$ for some constant K ; $\phi(x) = 0$ if and only if $x = 0$. For example, if (Ω, F, μ) is a measure space, and $0 < p < \infty$, then

$$\|f\| = \left(\int |f|^p \, d\mu\right)^{1/p}$$

defines a quasi-norm on $L^p(\mu)$. [It is a norm if $p \geq 1$.]

If E is a topological vector space whose topology is defined by a quasi-norm $\|\cdot\|$ that is uniformly continuous on bounded sets of E, then $(E, \|\cdot\|)$ is called a continuously quasi-normed space.

According to [6], a continuously quasi-normed space $(E, \|\cdot\|)$ is called locally PL-convex if and only if the function $\log\|\cdot\|$ is plurisubharmonic. Equivalently, the function $\|\cdot\|^p$ is plurisubharmonic for some (or all) p with $0 < p < \infty$, or the ball $\{x \in E : \|x\| < 1\}$ is pseudoconvex. Such a space is certainly locally pseudoconvex.

The most important class of examples for the present paper is the class of Banach spaces $(E, \|\cdot\|)$. Then $\|\cdot\|$ is a convex function, and therefore plurisubharmonic by Jensen's inequality.

A principle commonly used in (finite-dimensional) several complex variables says that a region is pseudoconvex if and only if it is "pseudoconvex at each boundary point". We state here an infinite-dimensional instance of this principle.

1.1 PROPOSITION. Let $(E, \|\cdot\|)$ be a continuously quasi-normed space. Then E is locally PL-convex if and only if, for every $x_o \in E$ with $\|x_o\| = 1$, there is a plurisubharmonic function ψ on E with $\psi(x_o) = 1$ and $\psi(x) \leq \|x\|$ for all x.

Proof. Suppose E is locally PL-convex, so that $\|\cdot\|$ is a plurisubharmonic function. Take $\psi(x) = \|x\|$. Conversely, suppose such functions ψ exist. We claim that $\|\cdot\|$ is plurisubharmonic. Let x_o, $y_o \in E$. If $x_o = 0$, then

$$\int_0^{2\pi} \|x_o + e^{i\theta} y_o\| \frac{d\theta}{2\pi} \geq 0 = \|x_o\|.$$

If $x_o \neq 0$, there is a plurisubharmonic function ψ with $\psi(x_o/\|x_o\|) = 1$

and $\psi(x) \leq \|x\|$ for all x, so

$$\int_0^{2\pi} \|x_o + e^{i\theta} y_o\| \frac{d\theta}{2\pi} = \|x_o\| \int_0^{2\pi} \| \frac{x_o}{\|x_o\|} + e^{i\theta} \frac{y_o}{\|x_o\|} \| \frac{d\theta}{2\pi}$$

$$\geq \|x_o\| \int_0^{2\pi} \psi(\frac{x_o}{\|x_o\|} + e^{i\theta} \frac{y_o}{\|x_o\|}) \frac{d\theta}{2\pi}$$

$$\geq \|x_o\| \; \psi(\frac{x_o}{\|x_o\|}) = \|x_o\| .$$

This shows that $\|\cdot\|$ is plurisubharmonic, and therefore that E is locally PL-convex.

The above Proposition may help explain the following definitions. Let $(E, \|\cdot\|)$ be a continuously quasi-normed space. Then $\|\cdot\|$ is <u>strictly plurisubharmonic</u> iff, for every $x_o \in E$ with $\|x_o\| = 1$ there exists a plurisubharmonic function ψ on E with $\psi(x_o) = 1$ and $\psi(x) < \|x\|$ for all $x \neq x_o$. Similarly, $\|\cdot\|$ is <u>uniformly plurisubharmonic</u> iff there is a continuous, increasing function $h : [0,\infty[\to [0,\infty[$, with $h(0) = 0$ and $h(t) > 0$ for $t > 0$, such that for every $x_o \in E$ with $\|x_o\| = 1$, there exists a plurisubharmonic function ψ on E with $\psi(x_o) = 1$ and $\psi(x) \leq \|x\| - h(\|x - x_o\|)$ for all $x \in E$.

The quasi-norm $(\int \|f\|^p d\mu)^{1/p}$ on $L^p(\mu)$ is uniformly plurisubharmonic, if $0 < p < \infty$. (See Section 3, and compare with [6] and [14].)

The reader may find it instructive to investigate whether replacing "plurisubharmonic" by "convex" yields conditions equivalent to strict convexity and uniform convexity.

The definitions above were inspired by a definition of Davis-Garling-Tomczak [6]. They say that E is <u>uniformly PL-convex</u> iff there is a function $h : [0,\infty[\to [0,\infty[$ with $h(t) > 0$ for $t > 0$, such that for all $x_o \in E$

with $\|x_o\| = 1$, and all $y \in E$,

$$\int_o^{2\pi} \|x_o + e^{i\theta}y\| \frac{d\theta}{2\pi} \geq 1 + h(\|y\|).$$

It is easy to see that this is true if $\|\cdot\|$ is uniformly plurisubharmonic. I do not know whether the converse is true; I expect that it is not.

The goal of this paper is to discuss convergence of martingales. The class of martingales must be restricted (to obtain convergence in L^1, for example). There are several possibilities for definitions; I have chosen one of them to use there. For simplicity, the discussion will be restricted primarily to the case of a separable Banach space E. We will write $PSH(E)$ for the set of all plurisubharmonic functions on E. Similarly, $PSH(V)$ is the set of all plurisubharmonic function on the open set V.

Let μ, υ be probability measures on the Borel subsets of E with first moment (i.e., $\int \|x\| d\mu(x) < \infty$, etc.). We say μ dominates υ, and write $\mu > \upsilon$ [PSH(E)], iff

$$\int \psi d\mu \geq \int \psi d\upsilon$$

for all $\psi \in PSH(E)$. For $x \in E$, we say μ is a Jensen measure for x, and write $\mu \sim x[PSH(E)]$ iff μ dominates the Dirac measure ε_x, that is,

$$\int \psi d\mu \geq \psi(x)$$

for all $\psi \in PSH(E)$. (See Gamelin [8] for discussion of Jensen measures in the finite-dimensional case.)

One good example of $\mu \sim x[PSH(E)]$ is obtained as follows. If $x, y \in E$, then the uniform distribution μ on the circle $x + e^{i\theta}y$, $0 \leq \theta < 2\pi$, represents x. That is,

$$\int_o^{2\pi} \psi(x + e^{i\theta}y) \frac{d\theta}{2\pi} \geq \psi(x)$$

for all $\psi \in PSH(E)$. Another example can be obtained as follows. Let υ be a probability measure in $\mathbb{C}$ and $\upsilon \sim 0$ [$PSH(\mathbb{C})$]. Suppose $f : \mathbb{C} \to E$ is holomorphic. Then the image measure $\mu = f(\upsilon)$ represents $f(0)$. The reason for this is that for any plurisubharmonic function ψ on E, the composition $\psi \circ f$ is subharmonic on $\mathbb{C}$.

If μ is a Jensen measure for x, then of course the barycenter of μ is x; i.e., $\int y\ \mu(dy) = x$. (If E is a separable Banach space, this exists as a Bochner integral.) The reason is that the real and imaginary parts of a linear functional are pluriharmonic, so we get

$$\int f(y)\,\mu(dy) = f(x)$$

for all $f \in E^*$. In general it is not enough to use only pluriharmonic functions in the definition of Jensen measures. Here is an example in one complex dimension, from [8]. The Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1-2r\cos\theta+r^2}$$

satisfies

$$P_0(\theta) = 1,$$
$$\min P_{1/2}(\theta) = \frac{1}{3},$$
$$\max P_{1/2}(\theta) = 3.$$

Let the measure μ on $\mathbb{C}$ be defined by

$$\mu = \frac{1}{3}\,\varepsilon_{1/2} + [1 - \frac{1}{3}\,P_{1/2}(\theta)]\lambda\ ,$$

where $d\lambda = d\theta/2\pi$ is Haar measure on the unit circle. Then μ is a probability measure. If f is harmonic on the closed disk, then

$$\int f d\mu = \frac{1}{3}f(\tfrac{1}{2}) + \int_0^{2\pi} f(e^{i\theta})[1 - \frac{1}{3}P_{1/2}(\theta)]\,\frac{d\theta}{2\pi}$$

$$= \frac{1}{3}\,f(\tfrac{1}{2}) + f(0) - \frac{1}{3}\,f(\tfrac{1}{2}) = f(0).$$

So μ represents 0 for all harmonic f. (In the language of Gamelin

[8], μ is an "Arens-Singer measure" for 0.) But μ is not a Jensen measure for 0. The function $\log|z|$ is subharmonic, but

$$\log|\tfrac{1}{2}| = \log|\tfrac{1}{2} - 0| \not\leq \int \log|\tfrac{1}{2} - z|\,\mu(dz) = -\infty .$$

Finally, we turn to the definitions concerning martingales. A discussion of real-scalar martingales in a Banach space can be found in [7, Chapter V].

Let (Ω,F,P) be a probability space. We will write $E[X] = \int X dP$ for the expectation if X is a random variable. Let $(F_n)_{n=0}^{\infty}$ be an increasing sequence of σ-algebras contained in F, and let E be a separable Banach space. If X_n is an F_n-Bochner integrable random variable, $X_n : \Omega \to E (n = 0,1,2,\ldots)$, then we will say that the sequence (X_n) is a PSH(E)-<u>martingale</u> iff, for every $\psi \in$ PSH(E), the real-valued process $(\psi(X_n))_{n=0}^{\infty}$ is a submartingale. Since the real part of a linear functional is plurisubharmonic, a PSH(E)-martingale is, in particular, a martingale in the usual sense. Another way to think of this is the following: Given X_n, the distribution of X_{n+1} is a measure that dominates the point X_n in the sense defined above. It follows that the image measures increase: $X_o(P) < X_1(P) < X_2(P) < \ldots$ [PSH(E)].

A useful class of examples can be found in [6]. Suppose $v_n : \Omega \to E$ is F_{n-1}-measurable $(n > 0)$ and v_o is constant in E; η_n is uniformly distributed on $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, η_n is independent of F_{n-1}, η_n is F_n-measurable. Then if $v_n \in L^p(\mu,E)$, the process

$$X_n = v_o + \sum_{k=1}^{n} \eta_k v_k$$

is called an H_p-<u>shrub</u> in [6] and called an <u>analytic</u> <u>martingale</u> in [2]. It can be seen that analytic martingales are PSH(E)-martingales. (The H_p-martingales defined in [6] are not used in this paper, since they need not converge, even in L^1.)

There is a connection with the <u>conformal</u> <u>martingales</u> of Getoor and Shape

[10]. If (X_t) is a martingale in E with continuous trajectories, then according to Proposition 5.9 of [16], (X_t) is a conformal martingale if and only if $(\psi(X_t))$ is a submartingale for any plurisubharmonic function where it makes sense. (Schwartz says these results "ne sont sans doute pas plus que des amusements".) But for discrete parameter martingales the situation is more complicated. In $\mathbb{C}$, there are martingales (X_n) such that (X_n^2) is also a martingale, but (X_n^3) is not a martingale. For example, $X_o = 0$, but $X_1 = X_2 = X_3 \ldots$ and

$$P[X_1 = 1] = \frac{1}{4}$$

$$P[X_1 = -1] = \frac{1}{4}$$

$$P[X_1 = i] = \frac{1}{4}$$

$$P[X_1 = -i] = \frac{1}{4}.$$

2. Martingale convergence

In this section it is proved that PSH(E)-martingales, bounded in $L^1(E)$-norm, converge a.s., provided the norm in E is uniformly plurisubharmonic. Some remarks on relaxing this condition are included in Section 4.

Throughout this section, $(E, \|\cdot\|)$ is a separable Banach space. Many of the assertions also hold in separable continuously quasi-normed spaces, or even non-separable spaces; but I will not spell that out here.

2.1 LEMMA. Suppose $\phi : E \to [-\infty, \infty[$ is upper semicontinuous and bounded above on bounded sets. Define $\psi_o(x) = \phi(x)$ and

$$\psi_{n+1}(x) = \inf \{ \int_o^{2\pi} \psi_n(x + e^{i\theta} v) \frac{d\theta}{2\pi} : v \in E \}$$

for $n \geq 0$. Then ψ_n decreases pointwise to the largest plurisubharmonic function less than or equal to ϕ.

Proof. Taking $v = 0$, we see that $\psi_{n+1}(x) \leq \psi_n(x)$, so $\psi(x) = \lim_{n\to\infty} \psi_n(x)$

exists for each $x \in E$. We will prove by induction that each ψ_n is upper semicontinuous. First, $\psi_o = \phi$ is upper semicontinuous. Suppose ψ_n is upper semicontinuous, and consider ψ_{n+1}. For fixed x, v the map $\theta \to \psi_n(x + e^{i\theta} v)$ is bounded above and upper semicontinuous, hence measurable. For fixed v, I claim that $x \to \int_o^{2\pi} \psi_n(x + e^{i\theta} v) \frac{d\theta}{2\pi}$ is upper semicontinuous. Indeed, if $x_k \to x$, then (by the upper semicontinuity of ψ_n) we have $\limsup_{k\to\infty} \psi_n(x_k + e^{i\theta} v) \leq \psi_n(x + e^{i\theta} v)$ for all θ. Now this is bounded above, so by Fatou's Lemma

$$\limsup_{k \to \infty} \int_o^{2\pi} \psi_n(x_k + e^{i\theta} v) \frac{d\theta}{2\pi} \leq \int_o^{2\pi} \limsup_{k \to \infty} \psi_n(x_k + e^{i\theta} v) \frac{d\theta}{2\pi}$$

$$\leq \int_o^{2\pi} \psi_n(x + e^{i\theta} v) \frac{d\theta}{2\pi} .$$

This shows that $x \to \int_o^{2\pi} \psi_n(x + e^{i\theta} v) \frac{d\theta}{2\pi}$ is upper semicontinuous. So ψ_{n+1} is the infimum of a family of upper semicontinuous functions, so it is upper semicontinuous. This completes the induction.

We next show that the limit ψ is plurisubharmonic. Fix x, v. Then by the monotone convergence theorem

$$\int_o^{2\pi} \psi(x + e^{i\theta} v) \frac{d\theta}{2\pi} = \lim_n \int_o^{2\pi} \psi_n(x + e^{i\theta} v) \frac{d\theta}{2\pi}$$

$$\geq \lim_n \psi_{n+1}(x) = \psi(x) .$$

So ψ is plurisubharmonic.

Next we show that ψ is the largest plurisubharmonic function $\leq \phi$. Suppose ψ' is any plurisubharmonic function $\leq \phi$. Then $\psi' \leq \psi_o = \phi$. By induction we see that $\psi' \leq \psi_n$ for all n, so $\psi' \leq \psi$.

The construction in this lemma can be rephrased in terms of PSH(E)-martingales. Suppose E is a separable Banach space, and ϕ, ψ_n, ψ are as in the Lemma. Then I claim that

$$\psi_n(x) = \inf E[\phi(x_n)] ,$$

where the infimum is over all analytic martingales $(X_k)_{k=0}^n$ with $X_o = x$. The proof is by induction on n. For $n=0$, we have only $X_o = x$, so $\psi_o(x) = \phi(x) = E[\phi(X_o)]$. Suppose the formula is known for $n-1$. Fix $x_o \in E$ and $\varepsilon > 0$. Choose v so that

$$\psi_n(x_o) \geq \int_o^{2\pi} \psi_{n-1}(x_o + e^{i\theta}v) \frac{d\theta}{2\pi} + \frac{\varepsilon}{2} .$$

Let X_1 have the uniform distribution on the circle $x_o + e^{i\theta}v$. Now choose measurably (by the Yankov-von Neumann selection theorem [12]) for each θ an analytic martingale $(X_k)_{k=1}^n$ with $X_1 = x_o + e^{i\theta}v$ and

$$\psi_{n-1}(x_o + e^{i\theta}v) \geq E[\phi(X_n)] + \frac{\varepsilon}{2} .$$

Thus we have obtained $(X_k)_{k=1}^n$ conditionally on X_1. Putting them together, we get $(X)_{k=0}^n$ with $X_o = x_o$, $E[\phi(X_n)] + \varepsilon \leq \psi_n(x_o)$.

<u>2.2 PROPOSITION</u>. Let E be a separable complex Banach space; let $0 < p < \infty$; let $h : [0,\infty[\to[0,\infty[$ be increasing and continuous. Then the following are equivalent:

(a) For every $x_o \in E$ with $\|x_o\| = 1$, there is a plurisubharmonic $\psi : E \to [-\infty,\infty[$ with $\psi(x_o) = 0$ and for all $x \in E$,

$$\psi(x) \leq \|x\|^p - 1 - h(\|x - x_o\|) .$$

(b) For every $x_o \in E$ with $\|x_o\| = 1$, and every analytic martingale (X_n) with $X_o = x_o$, we have

$$E[\|X_n\|^p] \geq 1 + E[h(\|X_n - x_o\|)] .$$

(c) For every $x_o \in E$ with $\|x_o\| = 1$, and every Borel measure γ on E representing x_o, we have

$$\int \|x\|^p \, d\gamma(x) \geq 1 + \int h(\|x - x_o\|) \, d\gamma(x) .$$

Proof. (c) $\Rightarrow$ (b). If (X_n) is an H_∞-shrub with $X_o = x_o$, then the distribution γ of X_n represents x_o.

(a) $\Rightarrow$ (c). Since ψ is plurisubharmonic, we have $\int \psi d\gamma \geq \psi(x_o)$ if $\gamma \sim x_o$ [PSH(E)]. Thus

$$\int \|x\|^p \, d\gamma(x) \geq \int (\psi(x) + 1 + h(\|x - x_o\|)) \, d\gamma(x)$$

$$\geq 0 + 1 + \int h(\|x - x_o\|) \, d\gamma(x).$$

(b) $\Rightarrow$ (a). Fix $x_o \in$ with $\|x_o\| = 1$. Define a function ψ_o by $\psi_o(x) = \|x\|^p - 1 - h(\|x - x_o\|)$. Then ψ_o is continuous and bounded on bounded sets. Let ψ be the largest plurisubharmonic function $\leq \psi_o$. It remains only to show $\psi(x_o) = 0$. Now $\psi = \lim\limits_n \psi_n$, where ψ is as in Lemma 2.1. Since $\psi_n(x_o) = 0$, it suffices to show $\psi_n(x_o) \geq 0$ for all n. But $\psi_n(x_o) = \inf E[\psi_o(X_n)]$, where the infimum is over all analytic martingales $(X_k)_{k=0}^n$ with $X_o = x_o$. By (b), we have $E[\psi_o(X_n)] = E[\|X_n\|^p - 1 - h(\|X_n - x_o\|)] \geq 0$.

According to this proposition, $\|\cdot\|$ is uniformly plurisubharmonic if and only if there is an h with $h(t) > 0$ for which (a), (b), (c) hold with $p=1$. It may be useful to note that "uniform PL-convexity" of [6] can be rephrased as follows: There is $h:[0,\infty[\to[0,\infty[$, increasing, continuous, $h(t) > 0$ for $t > 0$, such that for every $x_o \in E$, $\|x_o\| = 1$, and every analytic martingale $(X_k)_{k=0}^1$ with $X_o = x_o$, we have $E[\|X_1\|^p] \geq 1 + E[h(\|X_1 - x_o\|)]$. In fact, if this holds for one value of p $(0<p<\infty)$, then it holds for all.

The conditions in Proposition 2.2 are not changed if an exponent $1/p$ is added, as in the following.

2.3 PROPOSITION. Let $(E, \|\cdot\|)$ be a continuously quasi-normed space, and

$0 < p < \infty$. Then the following are equivalent:

(a) There is a function $h : [0, \infty[\to [0, \infty[$, increasing, continuous, $h(t) > 0$ for $t > 0$, such that for all $x_o \in E$ with $\|x_o\| = 1$ and all measures $\gamma \sim x_o$,

$$\int \|x\|^p d\gamma(x) \geq 1 + \int h(\|x - x_o\|) \, d\gamma(x) \; .$$

(b) There is a function $k : [0, \infty[\to [0, \infty[$, increasing, continuous, $k(t) > 0$ for $t > 0$, such that for all $x_o \in E$ with $\|x_o\| = 1$ and all measures $\gamma \sim x_o$,

$$\left(\int \|x\|^p \, d\gamma(x)\right)^{1/p} \geq 1 + \int k(\|x - x_o\|) \, d\gamma(x) \; .$$

Proof. If $p \geq 1$ and (a) holds, let $k(t) = (1 + h(t))^{1/p} - 1$. If $p \geq 1$ and (b) holds, let $h(t) = pk(t)$. If $p < 1$ and (a) holds, let $k(t) = ph(t)$. If $p < 1$ and (b) holds, let $h(t) = (1 + k(t))^p - 1$. The verifications, involving Jensen's inequality and Bernoulli's inequality, are omitted.

The following martingale convergence theorem is now easy to prove.

2.4 THEOREM. Let $(E, \|\cdot\|)$ be a Banach space; suppose $\|\cdot\|$ is uniformly plurisubharmonic. Let (X_n) be a PSH(E)-martingale; suppose $\sup E[\|X_n\|] < \infty$. Then X_n converges a.s.

Proof. Since all of the X_n are Bochner integrable, we may assume E is separable. We begin with (c) of Proposition 2.2: There is a function h with $h(t) > 0$ for $t > 0$ such that if $\|x_o\| = 1$ and $\gamma \sim x_o$ [PSH(E)], then

$$\int \|x\| d\gamma(x) \geq 1 + \int h(\|x - x_o\|) d\gamma(x) .$$

From this follows, for $x_o \in E$, $x_o \neq 0$, that if $\gamma \sim x_o$[PSH(E)], then

$$\int \|x\| d\gamma(x) \geq \|x_o\| + \|x_o\| \int h\left(\frac{\|x - x_o\|}{\|x_o\|}\right) d\gamma(x) \; .$$

Thus, if (X_n) is a PSH(E)-martingale, with $X_o = x_o$, we have

$$E[\|X_n\|] \geq \|x_o\| + \|x_o\| E[h(\frac{\|X_n - x_o\|}{\|x_o\|})] .$$

The conditional version of this implies that if (X_n) is a PSH(E)-martingale with respect to the σ-algebras (F_n), and if $n > m$, then

$$E[\|X_n\| \mid F_m] \geq \|X_m\| + \|X_m\| E[h(\frac{\|X_n - X_m\|}{\|X_m\|}) \mid F_m] .$$

Integrating this, we get

$$E[\|X_m\|] \geq E[\|X_m\|] + E[\|X_m\| h(\frac{\|X_n - X_m\|}{\|X_m\|})] .$$

Now $(X_n)_{n=0}^{\infty}$ is a PSH(E)-martingale, and $\|\cdot\|$ is plurisubharmonic, so $(\|X_n\|)_{n=o}^{\infty}$ is a submartingale, so it converges a.s., and $E[\|X_n\|]$ increases. Thus, as n and m increase without bound, we see that

$$E[\|X_n\|] - E[\|X_m\|] \to 0$$

so that

$$E[\|X_n\| \; h(\frac{\|X_n - X_m\|}{\|X_m\|})] \to 0 .$$

Thus $\|X_n\| \; h(\frac{\|X_n - X_m\|}{\|X_m\|})$ converges to zero in probability. But $\|X_n\|$ converges a.s., so $h(\frac{\|X_n - X_m\|}{\|X_m\|})$ converges to zero in probability (on the set where $\|X_n\|$ does not converge to zero), so $\|X_n - X_m\|$ converges to zero in probability. But (X_n) is a martingale, so in fact it converges a.e.

3. Integrable function spaces.

An interesting example of a space with uniformly plurisubharmonic norm is the space $L^1(\mu)$, where (Ω,F,μ) is a measure space. The first case is the one-dimensional space $\mathbb{C}$. The complex number 1 is the typical point on the surface of the unit ball. The function

$$h(t) = \frac{1}{16}\,\frac{t^2}{1+t}$$

satisfies $h(t) > 0$ for $t > 0$. The function

$$\psi(z) = \frac{1}{2}(\log|z| + \mathrm{Re}z - 1)$$

is subharmonic on $\mathbb{C}$. It is an elementary exercise to verify that

$$\psi(z) \leq |z| - 1 - h(|z - 1|)$$

for all $z \in \mathbb{C}$. So by Proposition 2.2, the absolute value $|\cdot|$ is "uniformly subharmonic".

So, if γ is a probability measure on $\mathbb{C}$ with $\gamma \sim 1$ $[PSH(\mathbb{C})]$, we have

$$\int |z|d\gamma(z) \geq 1 + \int h(|z - 1|)d\gamma(z).$$

More generally, if z_o is any complex number, and $\gamma \sim z_o$ $[PSH(\mathbb{C})]$, then

$$(*) \qquad \int |z|\, d\gamma(z) \geq |z_o| + |z_o| \int h\left(\frac{|z - z_o|}{|z_o|}\right)d\gamma(z).$$

In order to extend to function spaces $L^1(\mu)$, there are two possibilities. One is to start with $f_o \in L^1(\mu)$, $||f_o|| = 1$, and reduce to the case $f_o > 0$. Then

$$\psi(f) = \frac{1}{2}\int [f_o(\log|f| - \log|f_o|) + \mathrm{Re}f - \mathrm{Re}f_o]d\mu$$

is plurisubharmonic on $L^1(\mu)$ and satisfies

$$\psi(f) \leq \|f\| - 1 - h(\|f - f_o\|) .$$

The other possibility is to start with γ on $L^1(\mu)$, $\gamma \sim f_o$, then "disintegrate" it as $(\gamma_\omega)_{\omega \in \Omega}$, where $\gamma_\omega \sim f_o(\omega)$ for almost all ω. Then integrate the inequality (*) over Ω. The details are omitted.

4. Is there a complex Radon-Nikodym property?

For real Banach spaces, uniform convexity implies a martingale convergence theorem. But uniform convexity is a very strong condition; the condition that is both necessary and sufficient for martingale convergence (the Radon-Nikodym property) is interesting in its own right. This section contains some suggestions for a corresponding complex notion. However, the question is not answered here.

Let E be a complex Banach space. Under what conditions does every L^1-bounded PSH(E)-martingale converge? Is there a criterion like dentability (cf. [7, p. 133]), or a vector measure criterion (cf. [7, p. 127])? If C is a bounded subset of E, under what conditions does every PSH(E)-martingale with values in C converge? If C is open, it might be reasonable to consider PSH(C)-martingales, that is, sequences (X_n) such that $(\psi(X_n))$ is a submartingale for every plurisubharmonic function ψ defined on C.

Representability in the sense used here $(\mu \sim x[\mathrm{PSH}(E)])$ is more difficult to manage than the real version. This is illustrated in the following. The notation C_ε, where C is a set and $\varepsilon > 0$, will signify the ε-neighborhood $\{x \in E : \mathrm{dist}(x,C)<\varepsilon\}$ of C.

<u>4.1 PROPOSITION</u>. Let C be a Borel subset of the separable Banach space E, and let $x_o \in C$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5):

(1) There is a probability measure μ on C with $\mu \sim x_o$ [PSH(E)];

(2) There is no function $\psi \in \mathrm{PSH}(E)$ with $\psi \leq 0$ on C but $\psi(x_o)>0$;

(3) There is no function $\psi \in \mathrm{PSH}(E)$ with $\psi(x) \leq \mathrm{dist}(x,C)$ for all $x \in E$ and $\psi(x_o)>0$;

(4) For any $\varepsilon>0$, there is no analytic martingale $(X_k)_{k=0}^n$ with $X_o=x_o$ and $E[\mathrm{dist}(X_n,C)]<\varepsilon$;

(5) For any $\varepsilon, \varepsilon' > 0$, there is a probability measure $\mu \sim x_o[PSH(E)]$ with $\mu(C_\varepsilon) > 1 - \varepsilon'$.

Proof. (1) => (2). If $\psi \in PSH(E)$ and $\psi < 0$ on C, then $\psi(x_o) \leq \int_E \psi(x)\, d\mu(x) = \int_C \psi(x)\, d\mu(x) \leq 0$.

(2) => (3). If $x \in C$, then $\text{dist}(x,C) = 0$.

(3) => (4). Let $\psi_o(x) = \text{dist}(x,C)$. Let ψ be the largest plurisubharmonic function $\leq \psi_o$. By assumption, $\psi(x_o) \leq 0$. If the sequence ψ_n is defined as in Lemma 2.1, then there is n so that $\psi_n(x_o) < \varepsilon$. Therefore, there is an analytic martingale $(X_k)_{k=o}^n$ with $X_o = x_o$ and $E[\psi_o(X_n)] < \varepsilon$.

(4) => (3) is similar.

(4) => (5). Choose an analytic martingale $(X_k)_{k=o}^n$ with $X_o = x_o$ and $E[\text{dist}(X_n,C)] < \varepsilon\varepsilon'$. Let μ be the distribution of X_n, so $\mu(A) = P[X_n \in A]$ for Borel sets A. Then $\mu(C_\varepsilon) = P[d(X_n,C) < \varepsilon] > 1 - \varepsilon'$.

I do not know whether the conditions are equivalent under reasonable circumstances (such as C compact or C convex).

Martingale convergence leads to properties that resemble dentability. Here is one of the simplest such properties. (If "plurisubharmonic" is replaced by "convex", the conclusion would be that U is dentable.)

4.2 PROPOSITION. Let E be a separable Banach space. Suppose every L^1-bounded PSH(E)-martingale converges a.s. Let U be the open unit ball in E, and let $\varepsilon > 0$. Then there is a plurisubharmonic function ψ so that the set $\{x \in U : \psi(x) > 0\}$ is nonempty and has diameter less than ε.

Proof. Suppose there is $\varepsilon > 0$ so that if ψ is any plurisubharmonic function and $\psi(x) > 0$ somewhere on U, then diam $\{x \in U : \psi(x) > 0\} > \varepsilon$. We will construct a nonconvergent PSH(E)-martingale.

The probability space will be $\Omega = [0,1]^{\mathbb{N}}$ with P the product Lebesgue measure, and F_n the σ-algebra determined by the first n coordinates in Ω. (In fact [0,1] maps measurably onto any complete separable metric space, such as E, so this choice is unimportant.) Let $a_n = 1 - 2^{-n}$. There will be constructed sets $\Omega_n \in F_n$ with $P[\Omega_n] = 2^{-1} + 2^{-(n+1)}$, and random variables X_n with $E[\mathrm{dist}(X_n, a_{n-1}U)1_{\Omega_n}] < 2^{-(2n+2)}$, and $P\{\omega \in \Omega_n : \|X_n(\omega) - X_{n-1}(\omega)\| < \varepsilon/8\} < \varepsilon^{-1}2^{-(2n-1)}$.

Let $X_o = 0$, $\Omega_o = \Omega$. Suppose X_n has been constructed with $E[\mathrm{dist}(X_n, a_{n-1}U)1_{\Omega_n}] < 2^{-(2n+2)}$. Then $P\{\omega \in \Omega_n : \|X_n(\omega)\| > a_n\} < 2^{-(n+2)}$ since $a_n = a_{n-1} + 2^{-n}$. Choose $\Omega_{n+1} \subseteq \Omega_n \cap \{\|X_n\| \le a_n\}$ with $P[\Omega_{n+1}] = 2^{-1} + 2^{-(n+2)}$. On $\Omega \setminus \Omega_{n+1}$, let $X_{n+1} = X_n$. On Ω_{n+1}, proceed as follows. Suppose $X_n(\omega) = x$, where $\|x\| \le a_n$. Let $D = \{Y : \|y - x\| < \varepsilon/4\}$, so that $\mathrm{diam}(a_nU \cap D) < 2\ \varepsilon/4$, and hence $\mathrm{diam}\ (U \cap a_n^{-1}D) < \varepsilon a_n^{-1}/2 \le \varepsilon$. Thus if $\psi \in PSH(E)$ is ≤ 0 on $a_nU \setminus D$, then $\psi \le 0$ on a_nU, so $\psi(x) \le 0$. By Proposition 5, (2) = (4), there is a random variable Y representing x [PSH(E)] with $E[\mathrm{dist}(Y, a_nU \setminus D)] < 2^{-(2n+4)}$. Thus $E[\mathrm{dist}(Y, a_nU)] < 2^{-(2n+4)}$ and $P[\|Y - x\| < \varepsilon/8] < (8/\varepsilon)2^{-(2n+4)} = 2^{-(2n-1)}/\varepsilon$. The next step X_{n+1} will be chosen so that the conditional distribution, given $X_n = x$, is the distribution of Y. (These Y's should be chosen to depend measurably on x using the Yankov-von Neumann selection theorem.) So we get X_{n+1} with $X_{n+1} = X_n$ on $\Omega \setminus \Omega_{n+1}$, and $E[\mathrm{dist}(X_{n+1}, a_nU)1_{\Omega_{n+1}}] < 2^{-(2n+4)}$ and $P\{\omega \in \Omega_{n+1} : \|X_{n+1}(\omega) - X_n(\omega)\| < \varepsilon/8\} < 2^{-(2n-1)}/\varepsilon$. This completes the recursive construction of (X_n).

Now the PSH(E)-martingale (X_n) is L^1-bounded, since

$$E[\|X_n\|] = E[\sum_{k=0}^{n-1} \|X_k\| 1_{\Omega_k \setminus \Omega_{k+1}} + \|X_n\| 1_{\Omega_n}]$$

$$\le 1 + \sum_{k=1}^{n} E[\mathrm{dist}(X_k, U) 1_{\Omega_k}]$$

$$\le 1 + \sum_{k=1}^{n} 2^{-(2k+2)} < 2.$$

Let $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$. Then $P[\Omega_\infty] = 1/2$, but the series $\sum_{n=1}^{\infty} 2^{-(2n-2)}/\varepsilon$ converges, so a.s. on Ω_∞ we have $\|X_{n+1}(\omega) - X_n(\omega)\| \geq \varepsilon/8$ for all but finitely many n. Thus (X_n) does not converge a.s.

Notice that the martingale convergence is an isomorphic property, so the conclusion will hold also if U is the unit ball for an equivalent norm. I think it is unlikely that the converse is true. Another condition like dentability is given in the next result.

<u>4.3 PROPOSITION</u>. Suppose every L^1-bounded PSH(E)-martingale converges a.s. Let $\phi \in$ PSH(E) satisfy $\phi(X) \leq a\|X\| + b$, and let $\alpha \in \mathbb{R}$ satisfy $\inf \phi < \alpha < \sup \phi$. Then for any $\varepsilon > 0$, there is $\psi \in$ PSH(E) so that diam $\{x : \psi(x) > \phi(x)\} < \varepsilon$ and $\{x : \psi(x) > \phi(x),\ \alpha > \phi(x)\} \neq \emptyset$.

<u>Proof</u>. Suppose the conclusion is false. Then there is $\varepsilon > 0$ such that for all $\psi \in$ PSH(E), and all $x_o \in E$ such that $\phi(x_o) < \alpha$, if $\psi \leq \phi$ on $\{x : \|x - x_o\| > \varepsilon/2\}$ then $\psi \leq \phi$ everywhere.

Fix x_o with $\phi(x_o) < \alpha$, and let $\delta > 0$ be so small that $\phi(x_o) + \delta < \alpha$. Let $M = 1 + \sup\{\phi(x) : \|x - x_o\| < \varepsilon/2\}$. Define

$$\psi_o(x) = \begin{cases} M, & \text{if } \|x - x_o\| \leq \varepsilon/2 \\ \phi(x), & \text{if } \|x - x_o\| > \varepsilon/2 . \end{cases}$$

Now ϕ is upper semicontinuous and $M > \sup\{\phi(x) : \|x - x_o\| \leq \varepsilon/2\}$, it follows that ψ_o is upper semicontinuous. Define ψ_n as in Lemma 2.1, so that ψ_n decreases to ψ, the largest plurisubharmonic function $\leq \psi_o$. Now $\psi \leq \phi$ on $\{x : \|x - x_o\| > \varepsilon/2\}$, so $\psi \leq \phi$ everywhere. In particular, $\psi(x_o) \leq \phi(x_o)$. Now there is n so that $\psi_n(x_o) < \phi(x_o) + \delta$, so there is an anlytic martingale $(X_k)_{k=o}^{n}$ with $X_o = x_o$ and $E[\psi_o(X_n)] < \phi(x_o) + \delta$. Thus,

$$E[1_{\{\|X_n - x_o\| > \varepsilon/2\}}\, \phi(X_n)] + MP[\|X_n - x_n\| \leq \varepsilon/2] < \phi(x_o) + \delta .$$

Since $M = 1 + \sup\{\phi(x) : \|x - x_o\| \leq \varepsilon/2\}$, we get

$$E[1_{\{\|X_n - x_o\| > \varepsilon/2\}}\phi(X_n)] + E[1_{\{\|X_n - x_o\| \le \varepsilon/2\}}(X_n)] + P[\|X_n - x_o\| < \varepsilon/2]$$

$$< \phi(x_o) + \delta ,$$

so that

$$E[\phi(X_n)] + P[\|X_n - x_o\| \le \varepsilon/2] < \phi(x_o) + \delta .$$

But ϕ is plurisubharmonic, so $E[\phi(X_n)] \geq \phi(x_o)$. Thus $P[\|X_n - x_o\| \le \varepsilon/2] < \delta$ and $E[\phi(X_n)] - \phi(x_o) < \delta$. Therefore $E[\phi_n(X)] < \alpha$.

Using this construction, we will construct a martingale (Y_n) as before. Start at some point x_o with $\phi(x_o) < \alpha$. Choose δ_n decreasing rapidly to 0. We will get $\lim_n E[\phi(Y_n)] < \alpha$. If Y_n is defined, stop if $\phi(Y_n) \geq \alpha$, otherwise use the above to get Y_{n+1} with

$$P[\|Y_{n+1} - Y_n\| > \varepsilon/2 | \phi(Y_n) < \alpha] > 1 - \delta_n .$$

If δ_n converges to 0 fast enough, then $\lim_n E[\phi(Y_n)] < \alpha$, so (Y_n) is L^1-bounded, and we have $P[\lim_n \phi(Y_n) < \alpha] > 0$, and (Y_n) does not converge there.

This result, and others like it, may be more useful when stated in terms of functions in PSH(U) and PSH(U)-martingales, for some open set U, so that the martingale can be constructed inside the set U.

Here is one further remark. Fix $\eta > 0$, and U an open bounded set. If (X_n) is a PSH(U)-martingale, let

$$Q((X_n),\eta) = \{\omega : \|X_{n+1}(\omega) - X_n(\omega)\| < \eta \text{ except for finitely many } n\}.$$

Then let $\phi_\eta(x) = \inf P[Q((X_n),\eta)]$, where the inf is over all PSH(U)-martingales with $X_o = x$. The function ϕ_η is formally plurisubharmonic on U. Convergence of PSH(U)-martingales is related to whether $\phi_\eta(x)=1$ for all $\eta > 0$.

The paper [5] of Bukhvalov and Danilevich has recently been pointed out to me. It is concerned with the "analytic Radon-Nikodym property" for a complex Banach space E. It is characterized by the existence of boundary values for E-valued H^p-functions. It would be interesting to know whether there is a connection between the analytic Radon-Nikodym property and topics of this paper.

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